PROJECT REPORT OF SEMESTER -5 OF THE DEPARMENT OF MATHEMATICS (HONS.) .

Solving L.P.P by Moving Hyperplane Method

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Hyperplane :

Point sets are the sets whose elements are points or vectors in n-dimensional Euclidean space E^n. Thus the set

$$X = \{(x_1, x_2) : x_1^2 + x_2^2 < 1\}$$

represents set of points in E^2 lying inside a circle of unit radius with centre at the origin.

In two dimensions, a linear equation in x_1, x_2, of the form $c_1x_1 + c_2x_2 = z$ represents a **straight line.**

Similarly, in three dimensions, a linear equation in x_1, x_2, x_3, of the form $c_1x_1 + c_2x_2 + c_3x_3 = z$ represents a **plane.**

Thus a line is a set of points in E^2 satisfying $c_1x_1 + c_2x_2 = z$ and a plane is a set of points in E^3 satisfying $c_1x_1 + c_2x_2 + c_3x_3 = z$.

The above two equations can be written in a compact form as $cx = z$, where $c = (c_1, c_2)$ or (c_1, c_2, c_3) and $x = [x_1, x_2]$ or $[x_1, x_2, x_3]$ in two or three dimensions respectively. Generalising the idea of dimensions, we say that a set of points in n-dimensional space whose co-ordinates satisfy the linear equation of the form

$$c_1x_1 + c_2x_2 + \dots + c_nx_n = z$$

is called a **hyperplane** for fixed values of z and $c_i = 1, 2, \dots, n$.

The equation of the hyperplane can be put in short as $cx = z$ where $c = (c_1, c_2, \dots, c_n)$ and $x = [x_1, x_2 \dots, x_n]$ and in which all the c_i's are constants but not zero simultaneously. For different values of z, we get different hyperplanes. In the notation of sets,

$$H = \{ x : cx = z \}$$

is a **hyperplane** whose equation is $cx = z$.

NORMAL TO THE HYPERPLANE :

If $z = 0$, $cx = 0$, so that the hyperplane passes through the origin. From this we see that the vector c is **orthogonal** to every vector x on the hyperplane and this c is called the **normal to the hyperplane**.

If $z \neq 0$ and x_1, x_2, be two distinct points on the hyperplane $cx = z$. then

$$c(x_1 - x_2) = cx_1 - cx_2 = z - z = 0.$$

Thus c is **orthogonal** to any vector $(x_1 - x_2)$ on the hyperplane

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The two vectors $\pm\frac{c}{|c|}$ are **unit normals to the hyperplane.**

PARALLEL HYPERPLANES :

The hyperplanes having the same unit normals are said to be **parallel.** Moving a hyperplane cx = z parallel to itself is accomplished by increasing or decreasing the value of z

Open Half Spaces :

The hyperplane cx = z in E^n divides whole of E^n into three mutually exclusive and collectively exhausted disjoint sets as

$$X_1 = \{ x : cx < z \},$$
$$X_2 = \{ x : cx = z \}$$
$$\text{and} \quad X_3 = \{ x : cx > z \}.$$

X_1, and X_3, as defined above are called **open half spaces .**
The hyperplanes are closed sets .

Closed Half Spaces :

The hyperplane cx = z in E^n divides whole of E^n into three mutually exclusive and collectively exhausted disjoint sets as

$$X_1 = \{ x : cx < z \},$$
$$X_2 = \{ x : cx = z \}$$
$$\text{and} \quad X_3 = \{ x : cx > z \}.$$

sets like $X_4 = \{ x : cx \le z \}$ and $X_5 = \{ x : cx \ge z \}$ are called **closed half spaces .**
The hyperplanes are closed sets .

FOR EXAMPLE :

It is easily seen that the point x = [1, 2, 3, 4] lies in the open half space of the type cx > z generated by the hyperplane

$$2x_1 + 3x_2 + 4x_3 + 5x_4 = 7,$$

since 2.(1) + 3.(2) + 4.(3) + 5.(4) = 40 > 7.

But x= [1, 2, 3, -4] lies in the space cx < z.

NOTE : It should be noted that in a linear programming problem
Optimize z = cx subject to Ax (≤ = ≥) b, x ≥ 0,

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the objective function as also the constraints with the equality sign represent hyperplanes. The constraints with signs ≤ or ≥ are the half spaces produced by the hyperplanes with the sign of equality only.

Line Segment :

A line in the n-dimensional Euclidean space, passing through the points x_1 and x_2 ($x_1 \neq x_2$) is defined to be the set of points

$$X = \{ x : x = \lambda x_2 + (1 - \lambda)x_1 , \lambda \text{ is real} \}.$$

If the restriction $0 \leq \lambda \leq 1$ be imposed on λ, then the point x on this line is constrained to lie within the segment joining the points x_1 and x_2. Thus a set of points in the n-dimensional Euclidean space as given by

$$X = \{ x : x = \lambda x_2 + (1-\lambda) x_1 , 0 \leq \lambda \leq 1\}$$

is defined to be the **line segment** joining the points x_1 and x_2.

Hypersphere & Circle :

Consider a set of points

$$X = \{ x : |x-a| = \varepsilon > 0 \}.$$

This set of points forms a hypersphere in E^2 with centre at a and radius equal to ε.

If n = 2, then it is a circle in E^2 and
If n = 3, then it is a sphere in E^3.
An ε-neighbourhood about the point a is defined to be the set of points inside the hypersphere with centre a and radius $\varepsilon > 0$ and assumed to be very small.
Thus $|x - a| < \varepsilon$.

Interior Point & Boundary Point :

A point a is said to be an interior point of the set X if an ε-neighbourhood about the point a contains only points of the set X.

An interior point of X must be an element of X .

A point w is a boundary point of a set X if every ε-neighbourhood about w contains points of the set and also points not of the set. According to the definition, **it is clear that a boundary point may or may not belong to the set, but interior point must belong to the set.**

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A set is said to be closed if it contains the boundary points of the set. On the other hand, an open set contains only interior points of the set.

Thus $X = \{ (x_1, x_2) : x_1^2 + x_2^2 \leq 4 \}$ is a closed set
and $X = \{ (x_1, x_2) : x_1^2 + x_2^2 < 4 \}$ is an open set.
A set is said to be strictly bounded if there exists a positive number r such at for every $x \in X$. $|x| < r$. If each component of every point of a set has a lower limit, then the set is bounded from below.

Some Important Properties of Hyperplane :

i) HYPERPLANE IS A CONVEX SET .

Proof: Let us consider the hyperplane

$$X = \{ x : cx = z \}.$$

Let x_1 and x_2 be two points in X ; then

$$cx_1 = z \text{ and } cx_2 = z . \quad \text{.........(1)}$$

Now let the point x_3 be given by the convex combination of x_1 and x_2 as

$$x_3 = \lambda x_1 + (1-\lambda) x_2 ,\ 0 \leq \lambda \leq 1$$

Then

$$cx_3 = c\{ \lambda x_1 + (1-\lambda) x_2 \}$$

$$= \lambda cx_1 + (1-\lambda)cx_2$$

$$= \lambda z + (1 - \lambda) z, \qquad [\text{ by } (1)]$$

$$= z$$

so that x_3, satisfies $cx = z$. Thus x_3 is in X and it being the convex combination of x_1
and x_2 in X, X is a convex set.
Thus the hyperplane cx = z is a convex set.

ii) A HALF SPACE , OPEN OR CLOSED IS A CONVEX SET .

Proof: Let x_1 and x_2 be any two points of the closed half space ,

$$H_1 = \{ x : cx \geq z \}$$

Therefore , $cx_1 \geq z$ and $cx_2 \geq z$ ....(1)
If $0 \leq \lambda \leq 1$ then ,

$$c\{ \lambda x_1 + (1-\lambda) x_2 \}$$

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$= \lambda cx_1 + (1-\lambda)\, cx_2$
$\geq \lambda z + (1 - \lambda)\, z$ [by (1)]
$\geq z$

Hence $x_1, x_2 \in H_1$ and $0 \leq \lambda \leq 1$.
implies $[\lambda x_1 + (1-\lambda)\, x_2] \in H_1$ so H_1 is Convex .

Similarly Let x_1 and x_2 be any two points of the open half space , $H_2 = \{ x : cx \leq z \}$
If $x_1, x_2 \in H_2$ and $0 \leq \lambda \leq 1$.
Then replacing the inequality '≥' by '≤' in above it is true that $[\lambda x_1 + (1-\lambda)\, x_2] \in H_2$ So, H_2 is also Convex .

Now we are going to discuss about the process of solving L.P.P by moving hyperplane method and all its cases .

Solving L.P.P by Moving Hyperplane Method :

We alredy know that in a l.p.p the **objective function** and **constraints equations** represent so Solving L.P.P by Moving Hyperplane Method means Solving L.P.P by Graphical Method .

Graphical Method :

If the objective function be a function of two decision variables, then the problem can easily be solved graphically. In this method, we consider the inequations of the constraints as equations and draw the lines corresponding to these equations in a two dimensional plane and use the non-negativity restrictions. These lines define the region, in general a polygon, of permissible values of the variables as indicated by the inequations and equations of the constraints and the non-negativity relations. This permissible region for the values of the variables is called the ***feasible region*** or the ***solution space***. The first step in the graphical method is to plot the feasible solution space that satisfies all the constraints simultaneously. Then, by trial and error method, we find a point in this feasible region whose co-ordinates will give the optimal value (maximum or minimum) of the objective function. As will be seen in the examples, this point will be a corner (or vertex or extreme) point of the feasible region. Hence either the extreme points need be considered as candidates for the optimal solution or the point can be found by translating the straight line given by the objective function for some particular value of z, through this region. This will be explained by the following illustrative examples. This method is widely applied to problems with two decision variables. As the variables are constrained to be non-negative in all L. P. P., we need only examine the non-negative quadrant of the two dimensional space in graphical method.

FOR EXAMPLE :

Solve the following L. P. P. Graphically ,

Maximize $z = 150x + 100y$

subject to $8x + 5y \le 60,$

$4x + 5y \le 40, \; x, y \ge 0.$

Solⁿ :

The constraints are treated as equations along with the non-negativity relations. We confine ourselves to the non-negative quadrant of the xy-plane and draw the lines given by those equations.
Then the directions of the inequalities indicate that on the adjoining graph (Fig. 1) the region enclosed by

$8x + 5y = 60,$
$4x + 5y = 40,$
$x = 0 , y = 0$

will be the feasible region. All the points within this shaded region and on these lines will satisfy all the inequations.

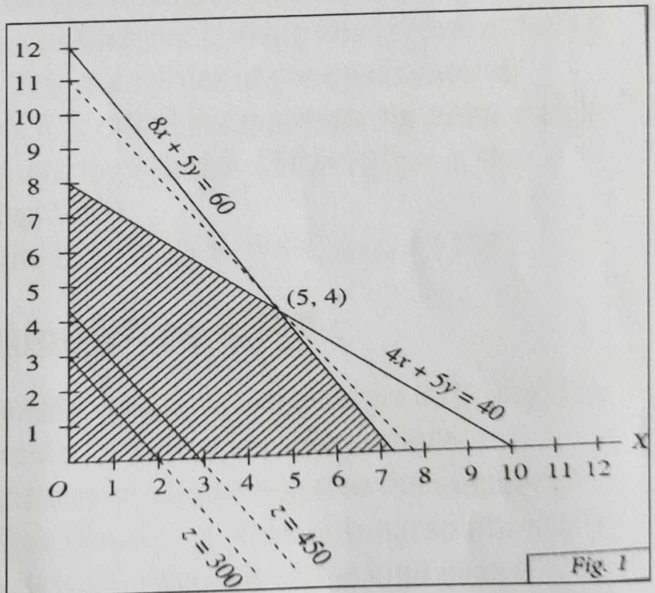

Fig. 1

For any particular value of z, the graph of the objective function regarded as an equation is a straight line and as z varies, a family of parallel lines is generated. A few of these lines are graphed for specific values of z and are shown in Fig. 1

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For z = 300, the objective function is 3x + 2y = 6.
For z = 450, the objective function is 3x + 2y = 9.
For z = 600, the objective function is 3x + 2y = 12.
For z = 900, the objective function is 3x + 2y = 18.
Considering the objective function as a straight line, called the profit line (in maximization problem), we see that the profit z is proportional to the perpendicular distance of this straight line from the origin. Hence the profit increases, as the profit line is translated away from the origin.
Our aim now is to find a point in this feasible region on the xy-plane which will give the maximum value of z. In order to find this point, we draw the profit line corresponding to some numerical value of z and move this line away from the origin, always being parallel to itself, until it contains only one point of the feasible region, whose corner points are (0, 0), (7.5, 0), (0, 8) and (5, 4).
We have drawn the first line corresponding to z = 300. Now, by translating this line, parallel to itself away from the origin over the shaded region, we find that the point with co-ordinates (5, 4) is the last point in the feasible region which the moving line (represented by the dotted line) encounters. This is a corner of the quadrilateral indicating the feasible region. It is called the ***maximizing point*** and by substituting its co-ordinates into the relation 150x+100y = z, the maximum profit is obtained as 1150.
Thus the optiml solution of the L.P.P is x = 5 , y = 4 , z_{max} = 1150 .

Nature of The Solution of an L.P.P :

A problem, having a single feasible solution, presents no difficulty. The possibility of its existence occurs if the number of equations (or inequations) in the constraints be at least equal to the number of variables. If the solution be feasible, then it is the optimal solution; if it is not, the problem has no solution. A problem of this kind even if there be a solution, is of no interest to us, since there is only one feasible solution and there is nothing to check for its optimality.

Thus a linear programming problem may have

(i) A UNIQUE AND FINITE OPTIMAL SOLUTION

(ii) AN INFINITE NUMBER OF OPTIMAL SOLUTIONS

(iii) AN UNBOUNDED SOLUTION

(iv) NO SOLUTION

(v) A UNIQUE SOLUTION FEASIBLE OR NOT .

Linear programming problems involving three variables can also be presented geometrically but their graphical solution is very difficult. The feasible region of solution in this case will be a three dimensional figure enclosed by the planes represented by the constraints regarded as equations. This figure is called a ***polyhedron***. The objective function in this case will represent a plane for certain value of z. The plane, representing the optimal value of z which has at least one point in common with the region of feasible solutions, gives the optimal value for the objective function. The point or points from the feasible region of solution which lie on the plane represented by the objective function for optimal z are optimal solutions.
The physical properties of the problems require that the values of the variables be integers. But this is not guaranteed except in special cases. The possible remedy in such cases is to round the continuous optimal solutions.

Examples of all Cases :

(i) A Unique and Finite Optimal Solution :

If we have a function to minimize rather than to maximize, in that case the cost line (in minimization problem) as given by the objective function is to be translated towards the origin. This case is given as an example below.

Minimize $z = 3x + 5y$

subject to $2x + 3y \geq 12$,

$-x + y \leq 3$,

$x \leq 4$ **and** $y \geq 23$.

Solⁿ :

In Fig. 2, the shaded region bounded by the constraint equations is the feasible region as indicated by the inequality signs. The corner points of the feasible region are $(\frac{3}{2}, 3)$, $(4, 3)$, $(4, 7)$ and $(\frac{3}{5}, \frac{18}{5})$. The cost line,

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as given by the objective function on the assumption of z = 30, is given by the dotted line in the figure. As this is the problem of minimization, the cost line is translated towards the origin and the cost function takes its minimum value at one corner point of the feasible region given by $x = \frac{3}{2}$, $y = 3$.

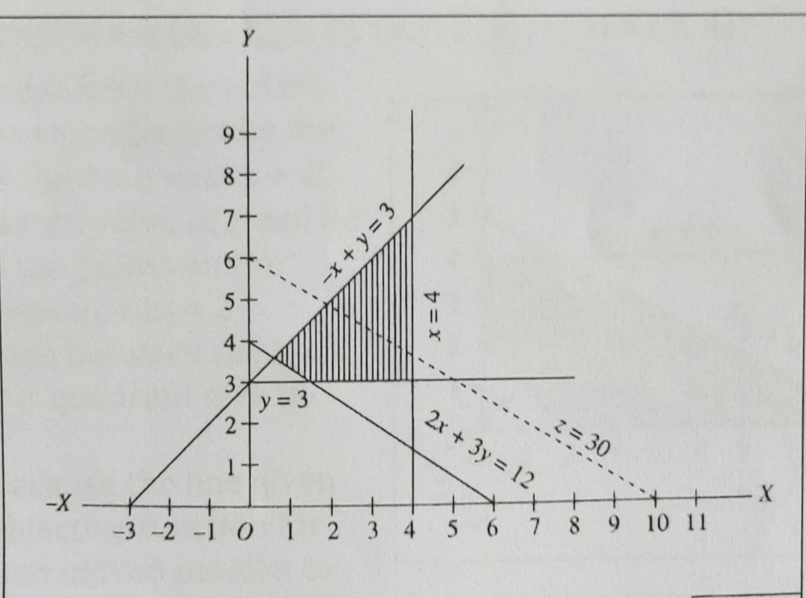

Fig. 2

The minimum value of the objective function is 19.5 there .
Thus the optimal solution of the L. P. P. is
$x = \frac{3}{2}$, $y = 3$ and $z_{min} = 19.5$

In the above two problems, we see that the L. P. P. has a ***unique optimal solution*** .

(ii) An Infinite Number of Optimal Solutions :

A special case occurs when the objective function for some particular value of z gives a straight line parallel to one given by one of the constraints. This case is discussed below with an example.

Minimize $z = x_1 + x_2$

subject to $5x_1 + 9x_2 \leq 45,$

$x_1 + x_2 \geq 2,$

$x_2 \leq 4$ **and** $x_1, x_2 \geq 0.$

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Solⁿ :

We draw the straight lines given by the constraints taken as equations. Then the directions of the inequalities and the non-negativity restrictions determine the region of the feasible solution as indicated in Fig. 3 by the shaded region. The corner points of the feasible region are (0, 2), (2, 0), (9, 0), ($\frac{9}{5}$, 4) and (0, 4).

As is evident from the nature of the line represented by the objective function with z = 4, the minimum value of z will be 2 and all the points on the straight line $x_1 + x_2 = 2$ intercepted between the axes in the first quadrant give this value.

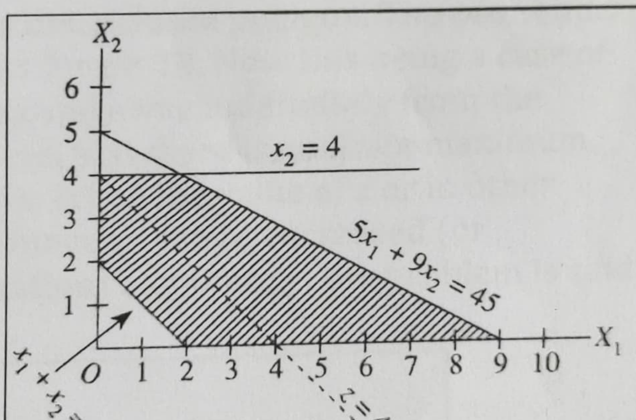

Fig. 3

This is because the line given by the objective function for z = 4 when moved parallel to itself towards the origin (as this is a case of minimization) coincides with the boundary line $x_1 + x_2 = 2$ of the feasible region and points corresponding to more than one corner of the feasible region give the optimum value. The problem has thus an infinite number of solutions (optimal) as any point on the line segment joining the corners intercepted by $x_1 + x_2 = 2$, $x_1 = 0$, $x_2 = 0$ is an optimal solution. Thus here $z_{min} = 2$, but the **number of solutions is infinite**. When such a situation exists, we say that there are ***alternative optimal solutions***.

(iii) An Unbounded Solution :

Sometimes it may so happen that the feasible region is unbounded in one direction and it does not form a polygon. In such cases, the line given by the objective function can be moved away indefinitely with a hope to contain only one point of the feasible region, but in vain.
The next example will present such a problem.

Maximize $z = 3x_1 + 4x_2$

Subject to $x_1 - x_2 \geq 0$

$-x_1 + 3x_2 \leq 3$

and $x_1 , x_2 \geq 0$

Soln : As before, we draw the straight lines given by the constraints considered as equations. The directions of the inequalities along with the non-negativity relations give the feasible region as shown. in Fig. 4. In this case, the feasible region is not a closed polygon. The profit line is shown in the diagram with dots for z = 12. Now this being a case of maximization, the profit line is moved away indefinitely from the origin parallel to itself and it is seen that there is no finite maximum value of within the feasible region. When the value of z or in other words the value of the objective function can be increased (or decreased in the case of minimization) indefinitely, the problem is said to have an ***unbounded solution*** .

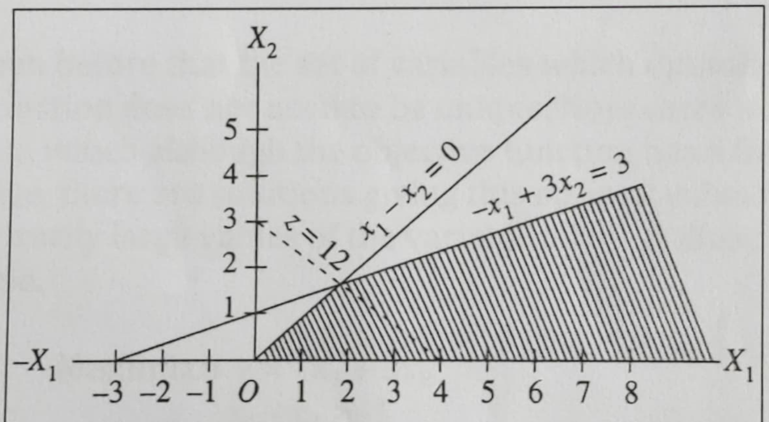

Fig. 4

NOTE : In the practical situation, it cannot be expected to get an L. P. P. with unbounded solution as this will imply the possibility of infinite profit or loss .

Unbounded feasible region does not necessarily imply that there will be no finite optimal solution of the problem as will be evident from the next example.

Maximize $z = 2x_1 - x_2$

subject to $x_1 - x_2 \geq 0$

$x_1 \leq 3$

and $x_1 , x_2 \geq 0$

Solⁿ :

In the adjoining diagram (Fig. 5), the feasible region for the solution of the problem is unbounded as given by the constraints but the profit line as represented by the dots for $z = 1$, when moved parallel to itself, away from the origin will contain the corner point (3, 2) only on it ultimately.
Thus the optimal solution is $x_1 = 3$, $x_2 = 2$ and the finite optimal value of the objective function is given by $z_{max} = 4$.

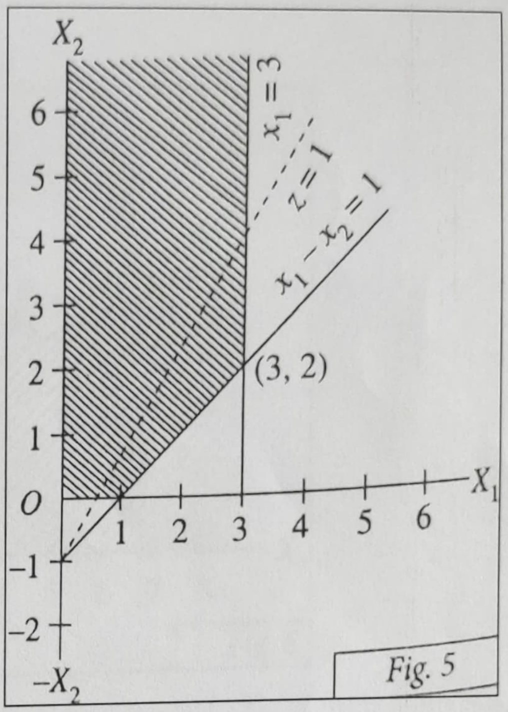

Fig. 5

We have seen before that the set of variables which optimizes the objective function does not need to be unique. Now cases may appear in practice in which although the objective function has a finite optimal value, there are solutions giving this optimal value for which we get arbitrarily large values of the variables. This is illustrated in the next example.

Maximize $z = -x_1 + 3x_2$

subject to $x_1 - x_2 \geq -1$

$-0.5x_1 + 1.5x_2 \leq 3$

and $x_1, x_2 \geq 0$

Solⁿ :

The dotted line in the diagram (Fig 6) gives the profit line corresponding to $z = 3$ and it is parallel to the edge of the infinite feasible region which is given by $-0.5x_1 + 1.5x_2 = 3$ and coincides with it when $z = 6$.
The maximum value of the objective function is thus 6. Again any point (x_1, x_2) lying on the edge of the feasible region (given by the constraint $0.5x_1 + 1.5x_2 \leq 3$) which extends to infinity, gives $z = 6$ and is therefore an optimal solution .

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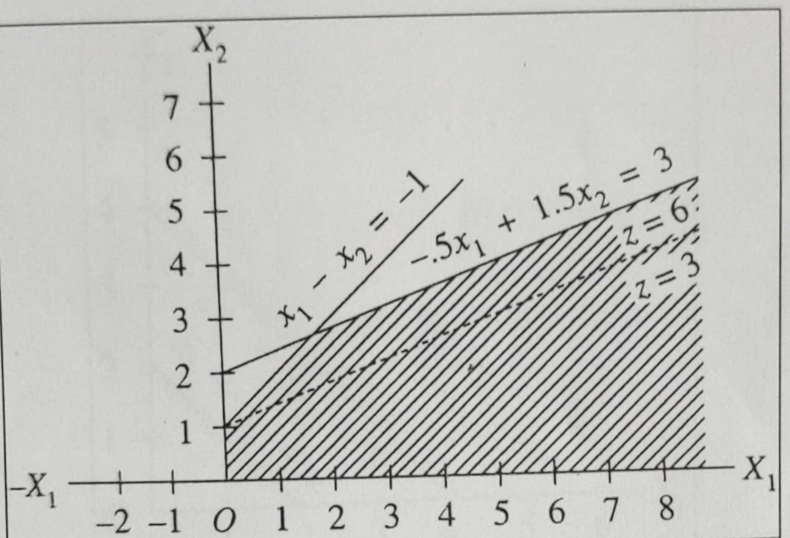

Fig. 6

The problem cannot be said to be completely in order as there are solutions with arbitrarily large values of the variables which give the optimal value of z.

(iv) No Feasible Solution :

So far we discussed problems for which we get a feasible region of solution. But there may be problems in which no feasible region of solution will be obtained. This happens when the given constraints are inconsistent. We illustrate this point below.

Maximize $z = 2x_1 - 3x_2$

subject to $x_1 + x_2 \le 2$

$2x_1 + 2x_2 \le 8$

and $x_1, x_2 \ge 0$

Solⁿ : As before, we draw straight lines given by the constraints considered as equations. The possible feasible region is shown in the diagram (Fig. 7) by the direction of the inequalities. As is obvious from the diagram due to the inconsistency of the constraints, no feasible solution is possible and hence no optimal solution.

The constraint set is empty in this case.

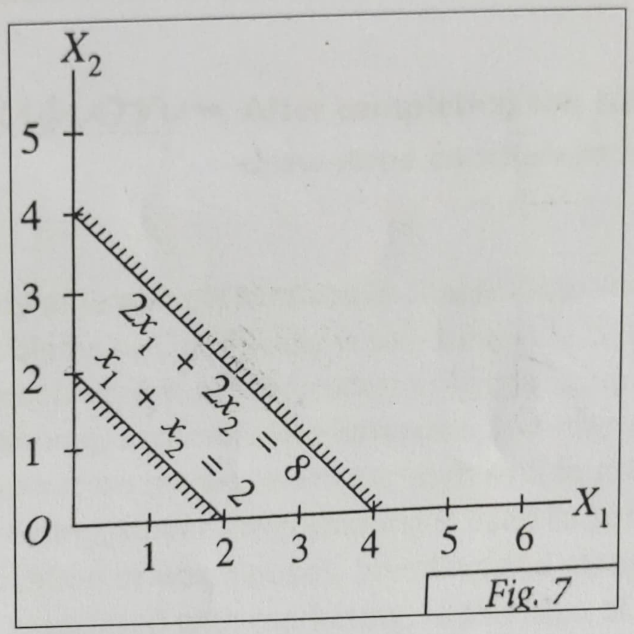

Fig. 7

There may be problems in which the constraints are consistent yet there may not be any feasible solution as no point satisfy simultaneously the constraints and the non-negativity restrictions. This is illustrated in the next example.

Maximize $z = x_1 + x_2$

subject to $x_1 - x_2 \geq 2$

$2x_1 - x_2 \leq -2$

and $x_1, x_2 \geq 0$

Sol^n^ : Any point in the shaded region satisfy the two constraints, but no point in the shaded region as given by the two constraints satisfy the non-negativity restrictions ($x_1, x_2 > 0$) and hence there is no feasible solution of the problem. (Fig. 8.)

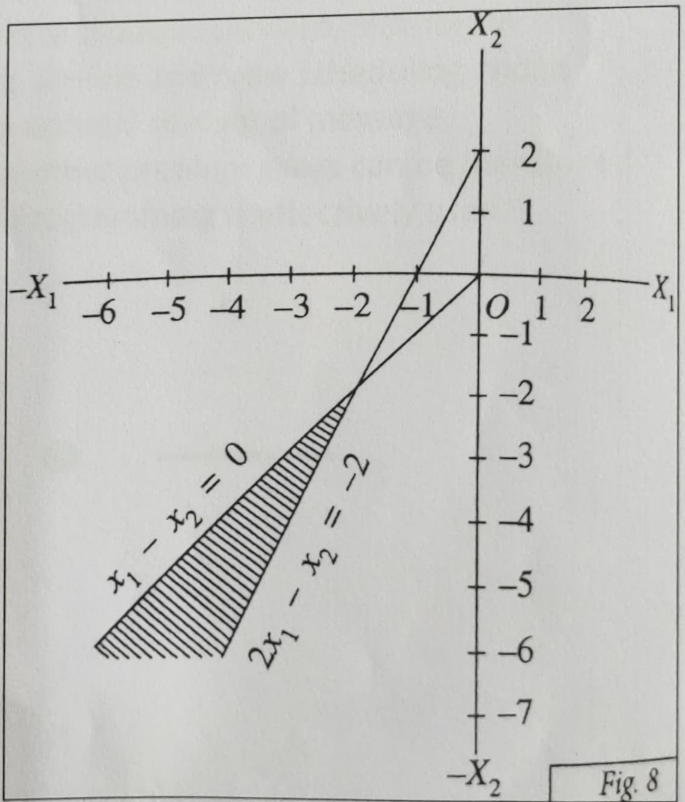

Fig. 8

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Significance of Graphical Method in L.P.P :

A Conclusion – After completing this tutorial we can draw some conclusions . They are as followings ,

- The potentiality of Graphical Method in Linear Programming as a tool for solving problems by Graphically is substantial. It is used to solve problems of procurement of raw materials in changing situations, production planning, assembly line balancing and many other problems of operation management through simple graphs. In the sphere of marketing, Linear Programming is used to solve problems of market mix, location of warehouses, blending and many other day to day problems associated with marketing. In the area of finance, Linear Programming technique is used in financing, profit planning and investment. In short it is an easiest visual representation of l.p.p .

- In addition to the above areas of its application, Linear Programming is used extensively in Government and public-services, diet-mix in hospitals, educational planning, air-line and crew scheduling and in food shipping plan. It is used in optimal routing of message communication network. Many other problem areas can be mentioned where the technique of Linear Programming is effectively used.